3D Ising Model
and
Conformal Bootstrap

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see also "*Lectures on CFT in D≥3*" @ sites.google.com/site/slavarychkov

Part 1
Conformal symmetry
(Physical foundations & Basics, Ising model as an example)

The subject of these lectures is:

The simplest

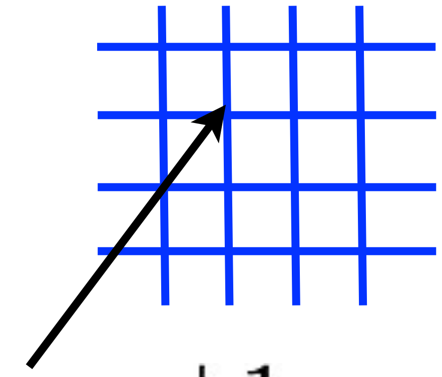
- experimentally relevant
- unsolved

Conformal Field Theory is 3D Ising Model @ T=Tc

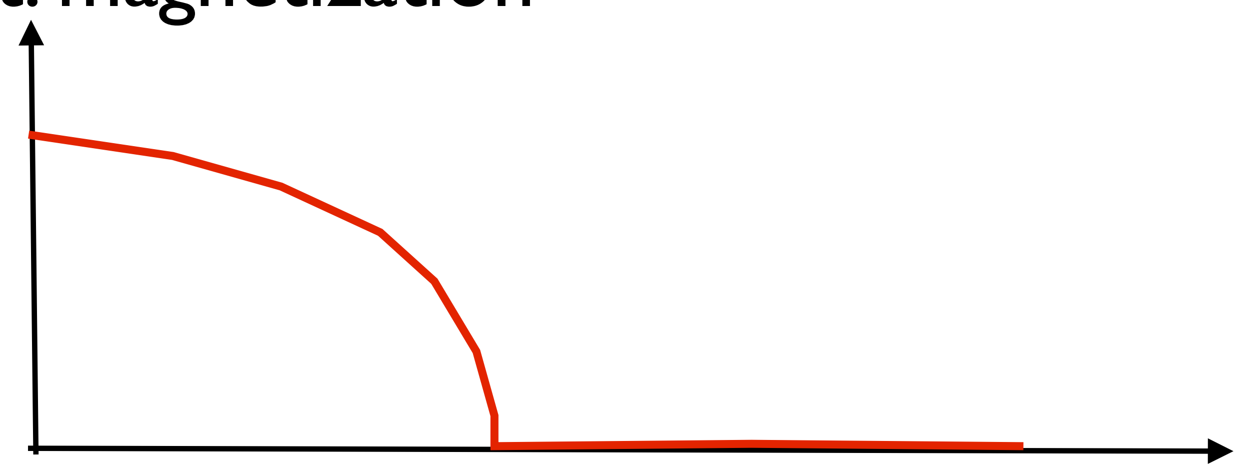
It's also an ideal playground to explain the technique of conformal bootstrap...

Basics on the Ising Model

$$Z = \exp\Big[\frac{1}{T}\sum_{\langle ij\rangle} s_i s_j\Big]$$



→ Paradigmatic model of ferromagnetism



Critical temperature (Curie point)

Correlation length

Critical point can also be detected by looking at the spin-spin correlations

For T>Tc : $\langle s(0)s(r)\rangle \sim e^{-r/\xi(T)}$ correlation length

$$\xi(T) \to \infty \quad (T \to T_c)$$

At T=Tc: $\langle s(0)s(r)\rangle \sim \frac{1}{|r|^{2\Delta}}$

Critical theory is scale invariant: $\langle s(0)s(\lambda r)\rangle = \lambda^{-2\Delta}\langle s(0)s(r)\rangle$

It is also conformally invariant [conjectured by Polyakov'71]

2D Ising Model

- free energy solved by Onsager'44 on the lattice and for any T

- Polyakov noticed that $\langle\sigma(x_1)\sigma(x_2)\epsilon(x_3)\rangle$ is conf. inv. at T=Tc

- In 1983 Belavin-Polyakov-Zamolodchikov identified the critical 2D Ising model with the first unitary minimal model

3D Ising Model

- Lattice model at generic T is probably not solvable

[many people tried]

- Critical theory (T=Tc) in the continuum limit might be solvable

[few people tried,
conformal invariance poorly used]

Existing approaches to 3D Ising

- Lattice Monte-Carlo

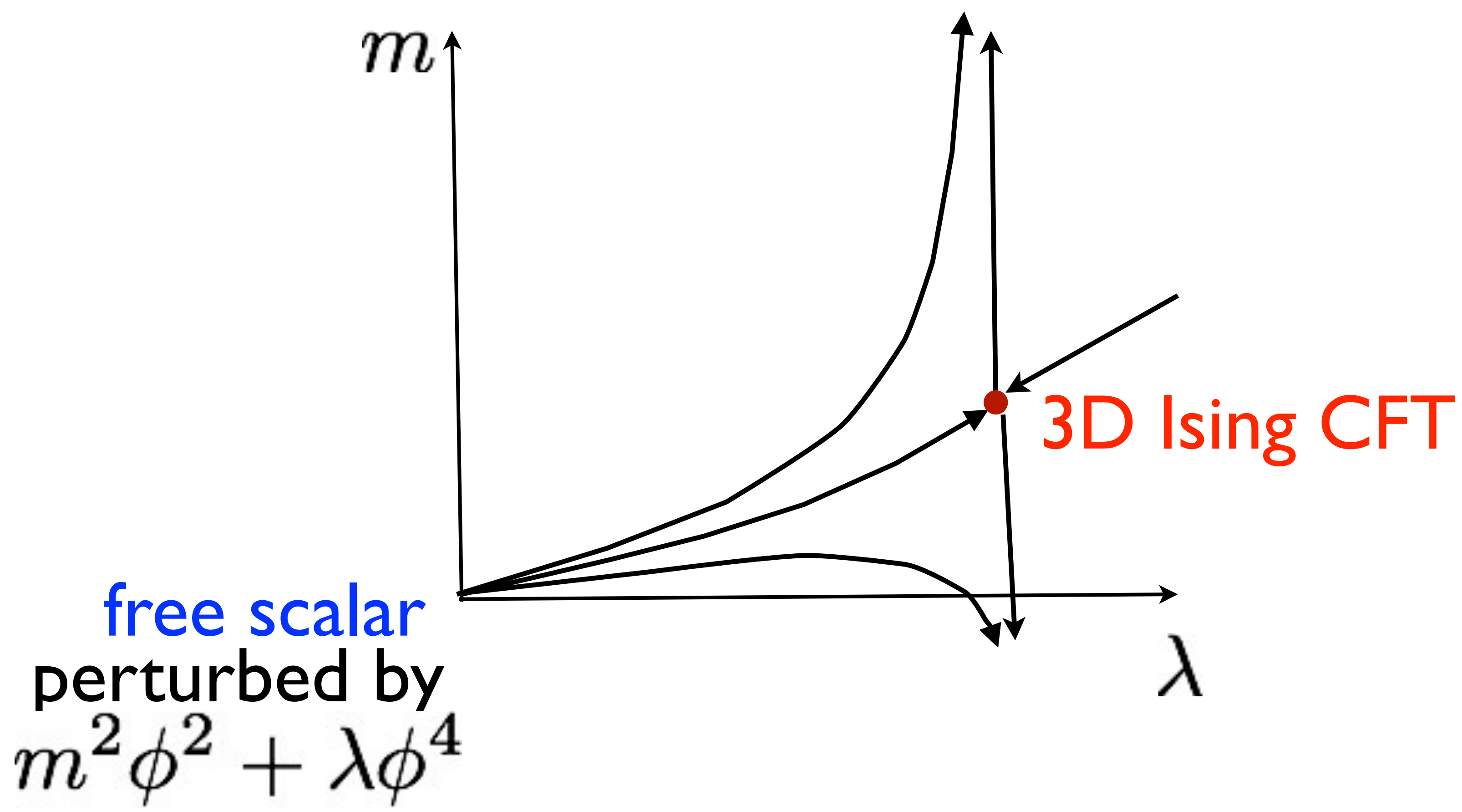
- High-T expansion on the lattice [~strong coupling expansion]

Expand exponential in $Z = \exp\left[\frac{1}{T}\sum_{\langle ij\rangle} s_i s_j\right]$

Converges for T>> Tc , extrapolate for T→ Tc by Pade etc

Existing approaches to 3D Ising

- RG methods



Physical Origins of CFT

RG Flows:

$$\mathrm{CFT}_{UV} \bullet\!\!\longrightarrow \mathrm{CFT}_{IR}$$

Fixed points = CFT

[Rough argument: $T^\mu_\mu = \beta(g)\mathcal{O} \to 0$ when $\beta(g) \to 0$]

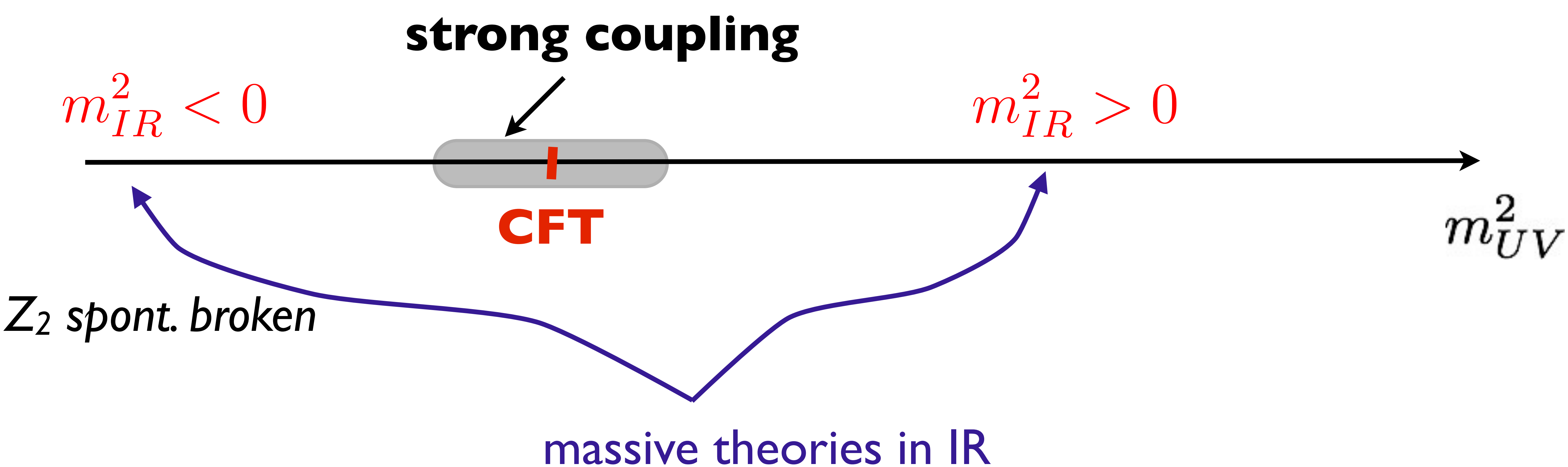
“Free to 3D Ising CFT” flow in more detail

CFT$_{UV}$ = free scalar $\quad (\partial\phi)^2$

Z_2-preserving perturbation: $\quad m^2\phi^2 + \lambda\phi^4\ [+\kappa\phi^6] \qquad m, \lambda \ll \Lambda_{UV}$

$$m_{IR}^2 = m_{UV}^2 + O\left(\frac{\lambda^2}{16\pi^2}\right)$$

IR physics phase diagram:



Universality

- Any same-symmetry Lagrangian (e.g. φ^6 coupling $\neq 0$) can flow to the same CFT_{IR}

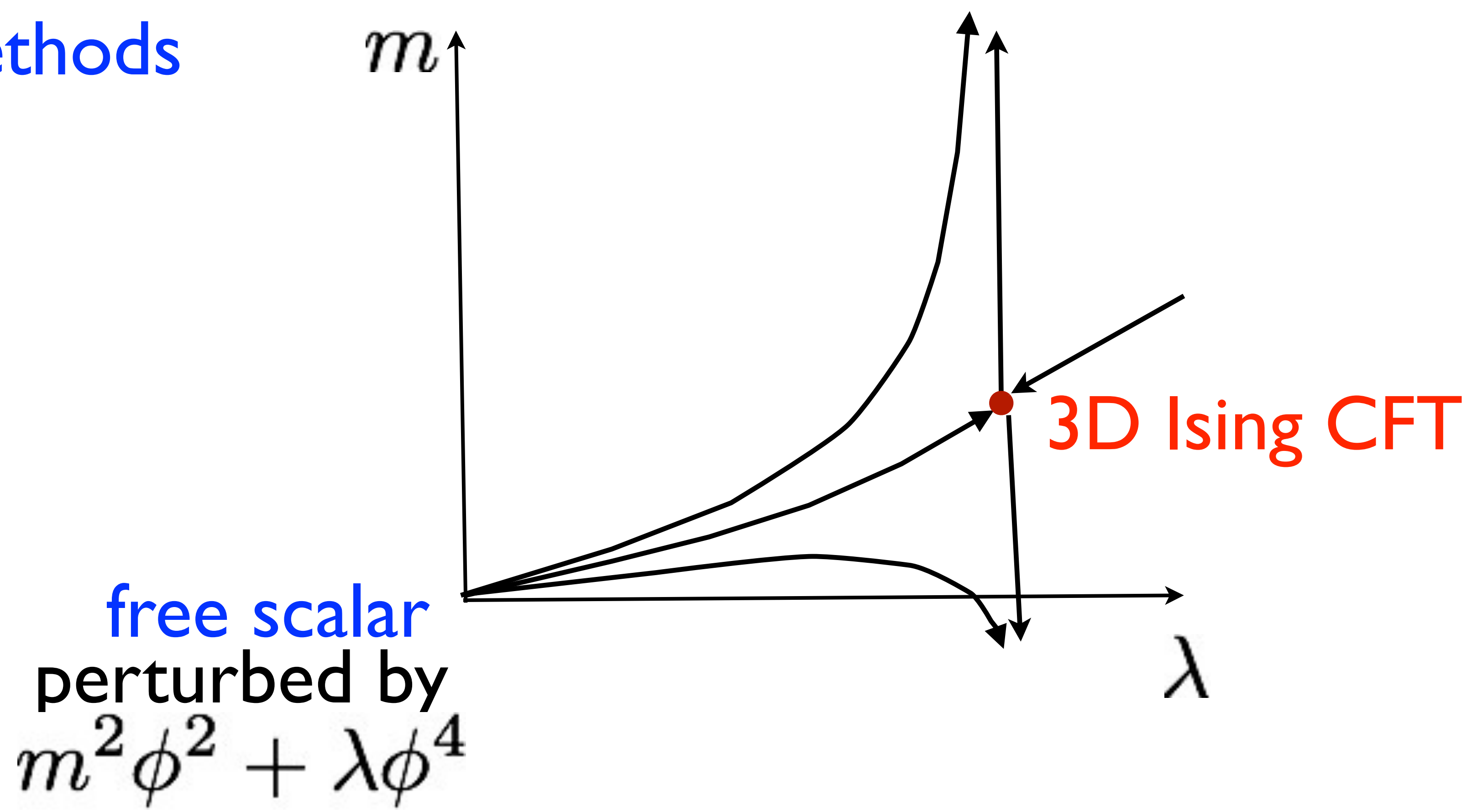
- In particular can even start from a lattice model like 3D Ising model or any modification (e.g. add next-to-nearest coupling)

Near T_c the spin-spin correlation length $\xi(T) \to \infty$
$\Rightarrow$ lattice artifacts go away

Continuum limit @ $T=T_c$ is the same CFT_{IR} as on the previous slide

Existing approaches to 3D Ising

- RG methods



Parisi... try to describe this fixed point by using renormalizable Lagrangian [?? since nonperturbative] + Borel resummation tricks

Wilson, Wegner… use Exact Renormalization Group [flow in the space of non-renormalizable Lagrangians]

Existing approaches to 3D Ising

- ε - expansion [Wilson, Fischer]

Fixed point becomes *weakly coupled* in 4-ε dimensions

⇒ compute all observables (e.g. operator dimensions) as power series in ε and set ε → 1 at the end

BUT: these series are divergent (starting from 2nd-3rd term) ⇒ Borel resummation etc needed

Existing approaches to 3D Ising

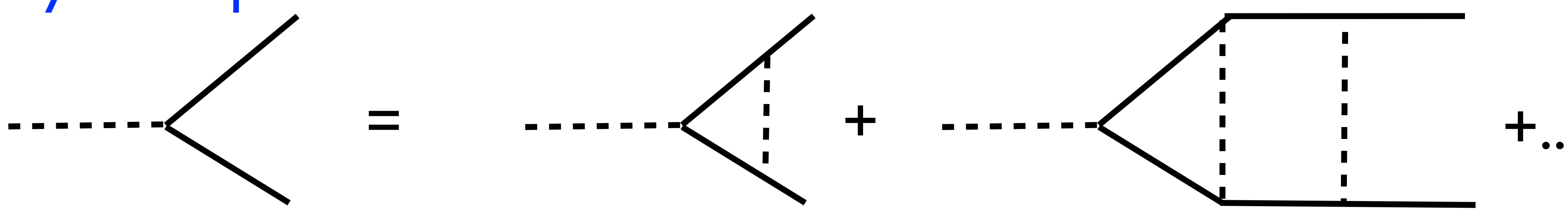
- skeleton expansion [Migdal; Parisi, Peliti...] for O(N) model at large N

The only old approach which tries to use conformal symmetry

$$\sigma_i, \epsilon : \qquad \Delta[\sigma_i] = 1/2 + O(1/N) \qquad\qquad \Delta[\epsilon] = 2 + O(1/N)$$

$$\langle \sigma\sigma\epsilon \rangle = \frac{f}{|x_{12}|^{2\Delta_\sigma - \Delta_\epsilon} |x_{13}|^{\Delta_\epsilon} |x_{23}|^{\Delta_\epsilon}} \qquad f = O(1/N)$$

Schwinger-Dyson equation for the vertex:



BUT: 1/N series converges very poorly even for N=3
(even worse than ε -expansion)

Conformal bootstrap conditioning

1. *Forget about Lagrangians -*
they are of little use for strongly coupled non-SUSY theories

2. *Forget about AdS -*
CFTs that we want to solve are non-SUSY, small-N and do not have AdS duals.

CFT - intrinsic definition

I. *Basis of local operators O_i with scaling dimensions Δ_i*

[including stress tensor $T_{\mu\nu}$ of Δ_T=4; conserved currents J_μ of Δ_J=3]

$$O_\Delta \stackrel{P}{\to} O_{\Delta+1} \stackrel{P}{\to} O_{\Delta+2} \stackrel{P}{\to} \dots$$

derivative operators **(descendants)**

K_μ = special conformal transformation generator, [K]=-1

$$K_\mu \leftrightarrow 2x_\mu(x\cdot\partial) - x^2\partial_\mu \qquad \text{cf.} \quad P_\mu \leftrightarrow \partial_\mu$$

$$O_\Delta \stackrel{K}{\leftarrow} O_{\Delta+1} \stackrel{K}{\leftarrow} O_{\Delta+2} \stackrel{K}{\leftarrow} \dots$$

In unitary theories dimensions have lower bounds:

$$\Delta \geq \ell + D - 2 \;(\geq D/2 - 1 \text{ for } \ell = 0)$$

So each multiplet must contain the lowest-dimension operator:

(primary)

$$K\mu \cdot O_\Delta(0) = 0$$

At x≠ 0: $$[K_\mu, \phi(x)] = (-i2x_\mu\Delta - 2x^\lambda\Sigma_{\lambda\mu} - i2x_\mu x^\rho\partial_\rho + ix^2\partial_\mu)\phi(x)$$

Ward identities for correlation functions:

$$X \cdot \langle \ldots \rangle = 0 \qquad X = (D, P_\mu, M_{\mu\nu}, K_\mu)$$

For 2- and 3-point functions suffice to solve the x-dependence:

normalization

$$\langle O_i(x) O_j(0) \rangle = \frac{\delta_{ij}}{(x^2)^{\Delta_i}}$$

$$\langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k} |x_{13}|^{\Delta_i+\Delta_k-\Delta_j} |x_{23}|^{\Delta_j+\Delta_k-\Delta_i}}$$

2. "coupling constants"

= OPE coefficients
= structure constants of the operator algebra

Notable operators of 3D Ising CFT

This CFT has an unbroken Z2 global symmetry [on the Lattice s→ -s]
All operators are Z2-odd (e.g spin field) or Z2-even (e.g. stress tensor)

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
σ	0	$-$	$0.5182(3)$	$\Delta = 1/2 + \eta/2$
σ'	0	$-$	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	$+$	$1.413(1)$	$\Delta = 3 - 1/\nu$
ε'	0	$+$	$3.84(4)$	$\Delta = 3 + \omega$
ε''	0	$+$	$4.67(11)$	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	$+$	3	n/a
$C_{\mu\nu\kappa\lambda}$	4	$+$	$5.0208(12)$	$\Delta = 3 + \omega_{\mathrm{NR}}$

- all are primaries of 3D conformal group

N.B. Primaries of D>=3 conformal group are morally similar to quasiprimaries of SL(2,C) in 2D (as opposed to Virasoro primaries)

σ = Spin field (lowest dimension Z2-odd scalar)

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[morally like φ with an anomalous dimension]

- We are in 3D, so $[\varphi_{free}]=0.5$, so the anomalous dimension ~0.02 is tiny.
- Anomalous dim. >0 because of the unitarity bounds
- In ε-expansion it equals $\varepsilon^2/108+O(\varepsilon^3)$

ε = "Energy density" field (lowest dimension Z2-*even* scalar)

(not to be confused with energy-momentum tensor)

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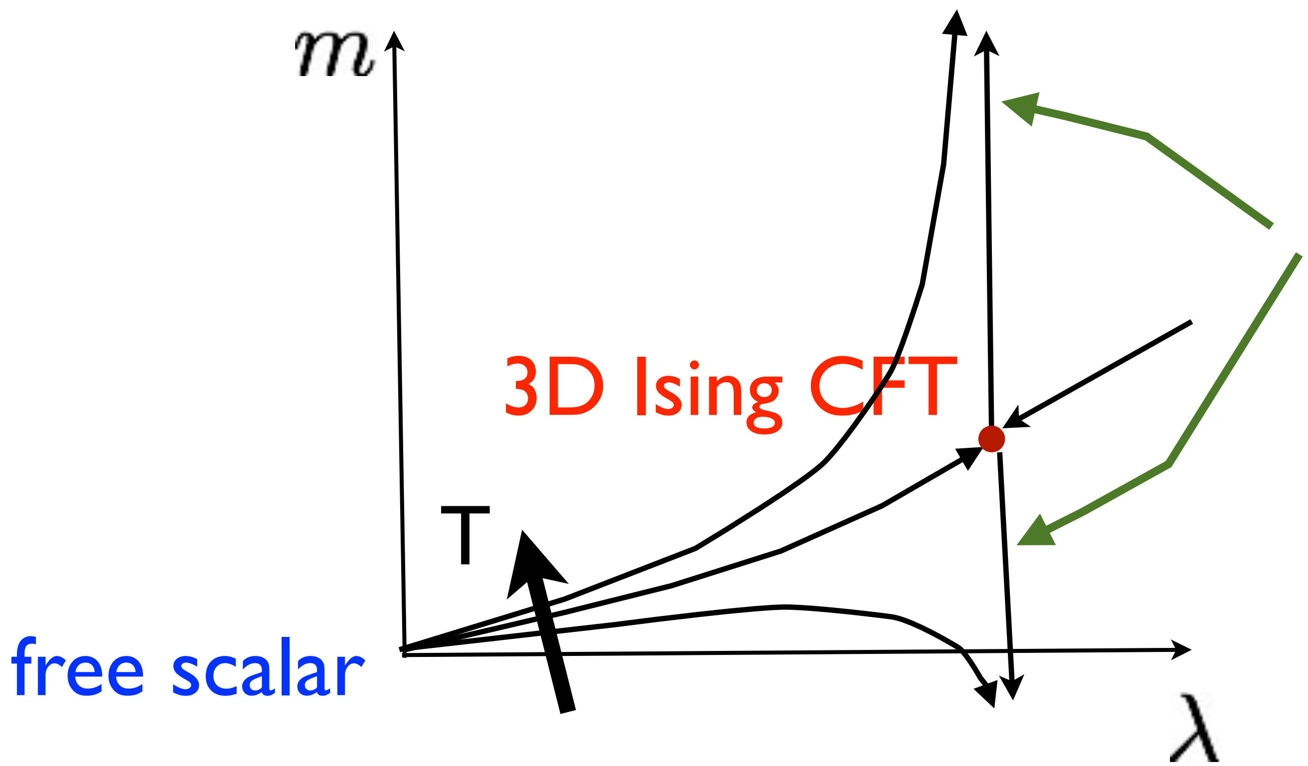
-like φ^2 but anomalous dimension ~0.4 nonnegligible

$-\varepsilon/3+O(\varepsilon^2)$ in the ε-expansion

Conformal perturbation theory



These flows can be described by:

$$\delta\mathcal{L}_{CFT} = \pm\mu^{3-\Delta_\epsilon}\int dx^3 \epsilon(x)$$

From the UV perspective: $$\mu^{3-\Delta_\epsilon} = \frac{\delta T}{T}\Lambda_{UV}^{3-\Delta_\epsilon}$$

Since correlation length $\xi(T) \sim \mu^{-1} \quad \Rightarrow \quad \xi(T) \sim \dfrac{1}{|T-T_c|^\nu}$

$$\nu = (3-\Delta_\epsilon)^{-1}$$

ε' = next-to-lowest Z2-even scalar

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
σ	0	$-$	0.5182(3)	$\Delta = 1/2 + \eta/2$
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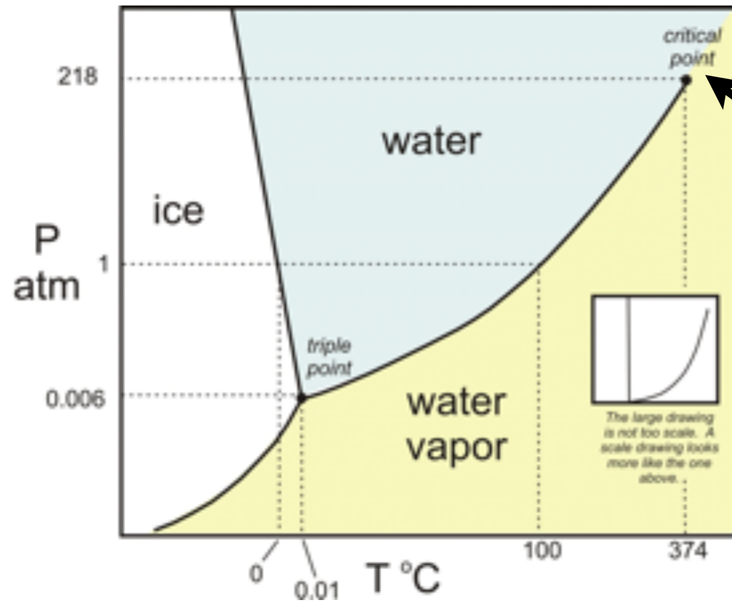
[~φ^4; relevant in free theory, becomes irrelevant in 3D Ising CFT]

$$\delta\mathcal{L} \sim \frac{1}{\Lambda^\omega}\int d^3x\, \epsilon'(x)$$ gives *corrections to scaling*

$$\langle\sigma(0)\sigma(x)\rangle = \frac{1}{|x|^{2\Delta_\sigma}}\left(1 + \frac{c}{(|x|\Lambda)^\omega}\right)$$

3D Ising CFT describes also liquid-vapor critical point:



emergent Z2 symmetry

To get to this point one has to finetune 2 parameters: P,T =
the total number of relevant scalars (one Z2-even and one Z2-odd)

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Stress tensor

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
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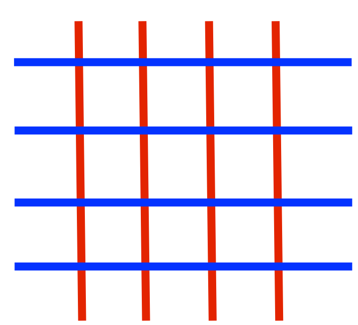
$$\delta\mathcal{L}_{CFT} = \alpha \int T_{xx}$$

is the continuum description of introducing spin coupling anisotropy:



$$J_x \neq J_y = J_z$$

Spin 4 symmetric traceless (not conserved!)

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
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describes effects of rotational symmetry breaking on cubic lattice:

$$\delta\mathcal{L}_{\mathrm{CFT}} \propto C_{1111} + C_{2222} + C_{3333}$$

Cf. rigorous CFT theorems.I

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
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- Unitarity bounds [Ferrara, Gatto, Grillo'74][Mack'77]

$$\Delta \geq D/2 - 1 \quad (l = 0)\,, \qquad \Delta \geq l + D - 2 \quad (l \geq 1)$$

- CFT "Coleman-Mandula" theorem [Maldacena,Zhiboedov'2011]

No conserved higher spin currents

Cf. rigorous CFT theorems.2

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
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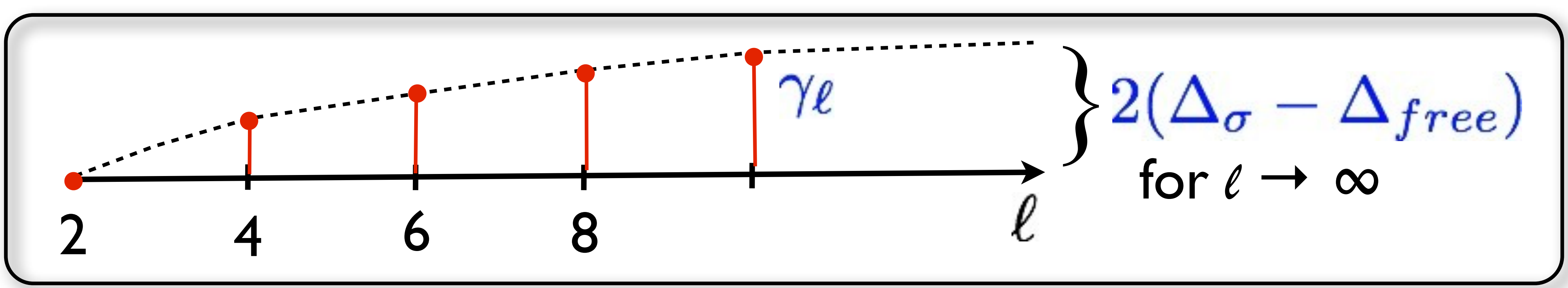
Sequence of anomalous dimensions of leading currents in σ x σ OPE

$$\gamma_\ell = \Delta_{\ell,\min} - \Delta_{cons}.$$

(a) is increasing & upward convex [Nachtmann'1973]

(b) tends to $2(\Delta_\sigma - \Delta_{free})$ for $\ell \to \infty$ [Callan, Gross'1973] [Alday,Maldacena 2007] [Fitzpatrick,Kaplan,Poland,Simmons-Duffin'12] [Komargodski,Zhiboedov'12]



Operator Product Expansion

$$O_i(x)O_j(0) = \lambda_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\{O_k(0)+\dots\}$$

$$\frac{1}{2}x^\mu\partial_\mu O_k + \alpha x^\mu x^\nu \partial_\mu\partial_\nu O_k + \beta x^2\partial^2 O_k + \dots$$

coefficients are fixed by conformal symmetry;
can be determined by plugging OPE into 3-point function and matching on the exact expression:

$$\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{13}|^{\Delta_i+\Delta_k-\Delta_j}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}}$$

E.g. for $\Delta_i = \Delta_j,\ \Delta_k \equiv \Delta$

$$\alpha = \frac{\Delta+2}{8(\Delta+1)}, \qquad \beta = -\frac{\Delta}{16(\Delta - D/2+1)(\Delta+1)}$$

OPE structure of 3D Ising

odd x odd = even
odd x even = odd
even x even = even

$$\sigma \times \sigma = 1 + (\epsilon + \epsilon' + \ldots) \\ + (T_{\mu\nu} + \ldots) \\ + (C_{\mu\nu\kappa\lambda} + \ldots) \\ + \ldots$$

even spins only
(next slide)

$$\sigma \times \epsilon = \sigma + \sigma' + \ldots \\ + \ldots$$

all spins

Bose symmetry in CFT

OPE of two **identical** scalars contains only **even spin** primaries

Consider three point function <scalar-scalar-spin l>

$$\langle \phi(x)\phi(-x)\mathcal{O}_{\mu_1,\mu_2,\ldots\mu_l}(0)\rangle \propto x_{\mu_1}x_{\mu_2}\ldots x_{\mu_l}$$

a) Since should be invariant under x→ -x, so vanishes for odd spin
b) By conformal invariance vanishes at any other three points ⇒

OPE coefficient vanishes

Remarks

1) Notice that odd spin descendants of course do occur in the OPE)

2) By a similar argument **antisymmetric** tensor fields cannot occur in the OPE of two scalars, **identical or not**]

Part 2.
Conformal Bootstrap. Theory

Intrinsic definition of CFT - recap

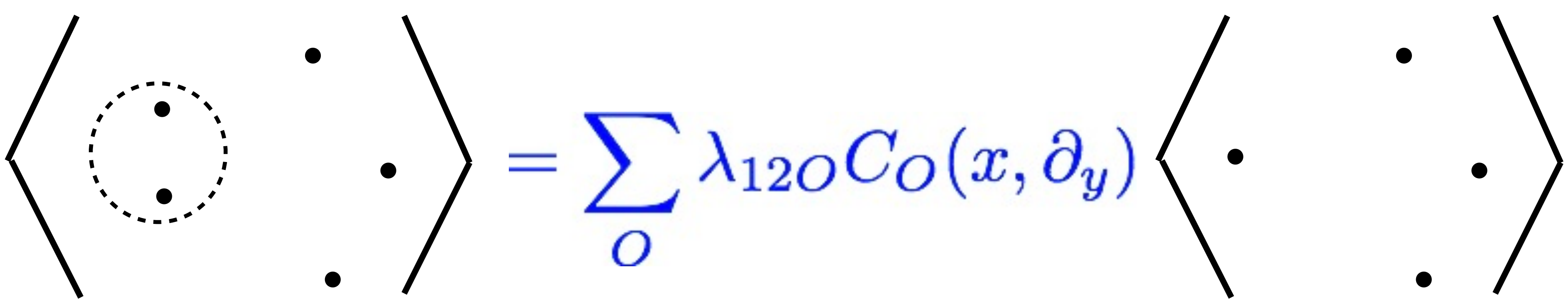
Any CFT is characterized by **CFT data**

- spectrum of primary operator dimensions and spins $\{\Delta_i, \ell_i\}$
- OPE coefficients f_{ijk}

$$O_i(x)O_j(y) = \sum f_{ijk} C(x-y, \partial_y) O_k(y)$$

fixed by conformal symmetry

Using OPE, any n-point function can be computed reducing to (n-1)-point functions:



And eventually to 2-pt functions which are known: $\langle O_i(x) O_j(0) \rangle = \dfrac{\delta_{ij}}{|x|^{2\Delta_i}}$

For 3-point functions we get:

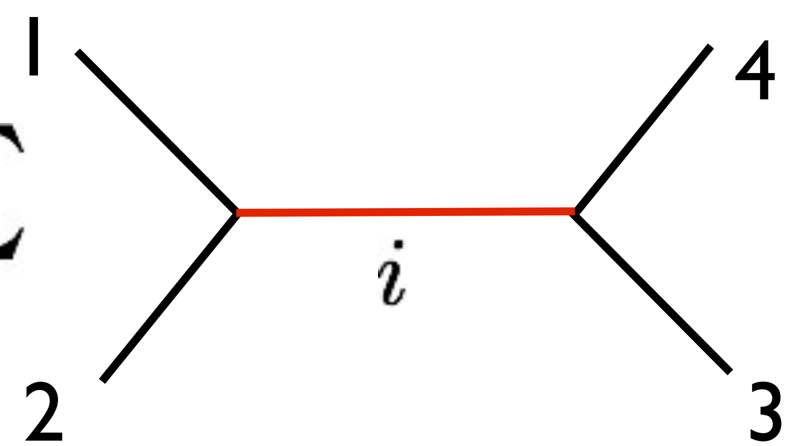
$$\langle O_i(x_1)O_j(x_2)O_k(x_3)\rangle = \frac{f_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{13}|^{\Delta_i+\Delta_k-\Delta_j}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}}$$

independently of which pair of operators is replaced by OPE

For 4-point functions we get:

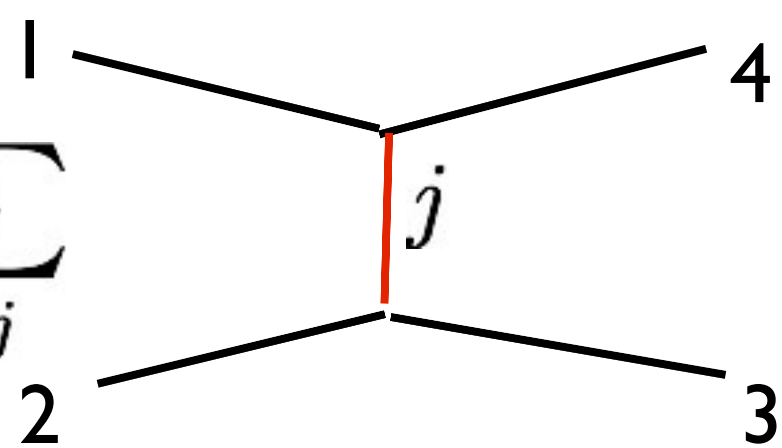
$$\underbrace{\langle O_1 O_2}\underbrace{O_3 O_4\rangle} = \sum_i f_{12i} f_{34i} \underbrace{C(x_{12}, \partial_2) C(x_{34}, \partial_4) \langle O_i(x_2) O_i(x_4)\rangle}_{\text{conformal partial wave}}$$

$$\sum_i$$



However we can also apply OPE in another channel:

$$\sum_j$$



$$\sum_j f_{14j} f_{23j} [\ldots]$$

The case of four identical scalars

Ward identity constrains 4-point function to have the form:

$$\langle\phi\phi\phi\phi\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}$$

$$\boxed{u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}}$$

Using OPE can say more:

$$\left\langle \begin{matrix} \phi(x_1) & \phi(x_3) \\ \phi(x_2) & \phi(x_4) \end{matrix} \right\rangle = \sum \lambda_{\phi\phi i}^2 |x_{12}|^{\Delta_i - 2\Delta_\phi} |x_{34}|^{\Delta_i - 2\Delta_\phi} \langle \{O_i(x_2) + \cdots\}\{O_i(x_4) + \cdots\}\rangle$$

$$= \sum \lambda_{\phi\phi i}^2 \frac{G_{\Delta_i,\ell_i}(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}$$

conformal blocks

$$\boxed{g(u,v) = 1 + \sum \lambda_{\phi\phi i}^2 G_{\Delta_i,\ell_i}(u,v)}$$

contribution of the unit operator

Crossing symmetry

$$x_1 \leftrightarrow x_3$$

$$\langle\phi\phi\phi\phi\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}} = \frac{g(v,u)}{|x_{14}|^{2\Delta}|x_{23}|^{2\Delta}}$$

$$\boxed{u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}}$$

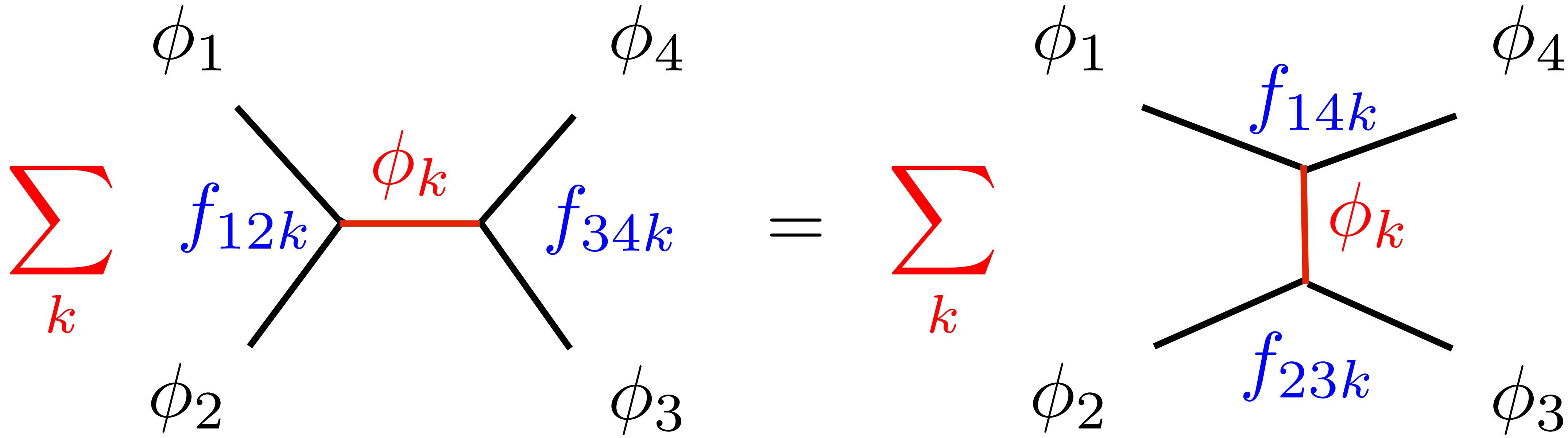
So: $\quad g(v,u) = (v/u)^{\Delta_\phi} g(u,v)$

This is a consistency condition for the CFT data

Nontrivial because not satisfied term by term in the expansion

$$g_s(u,v) = 1 + \sum \lambda_{\phi\phi i}^2 G_{\Delta_i,\ell_i}(u,v)$$

Crossing symmetry/OPE associativity/conformal bootstrap



- The agreement is **not automatic** ⇒ constraint on CFT data

[Ferrara,Gatto,Grillo 1973]
[Polyakov'74] [Mack'77]

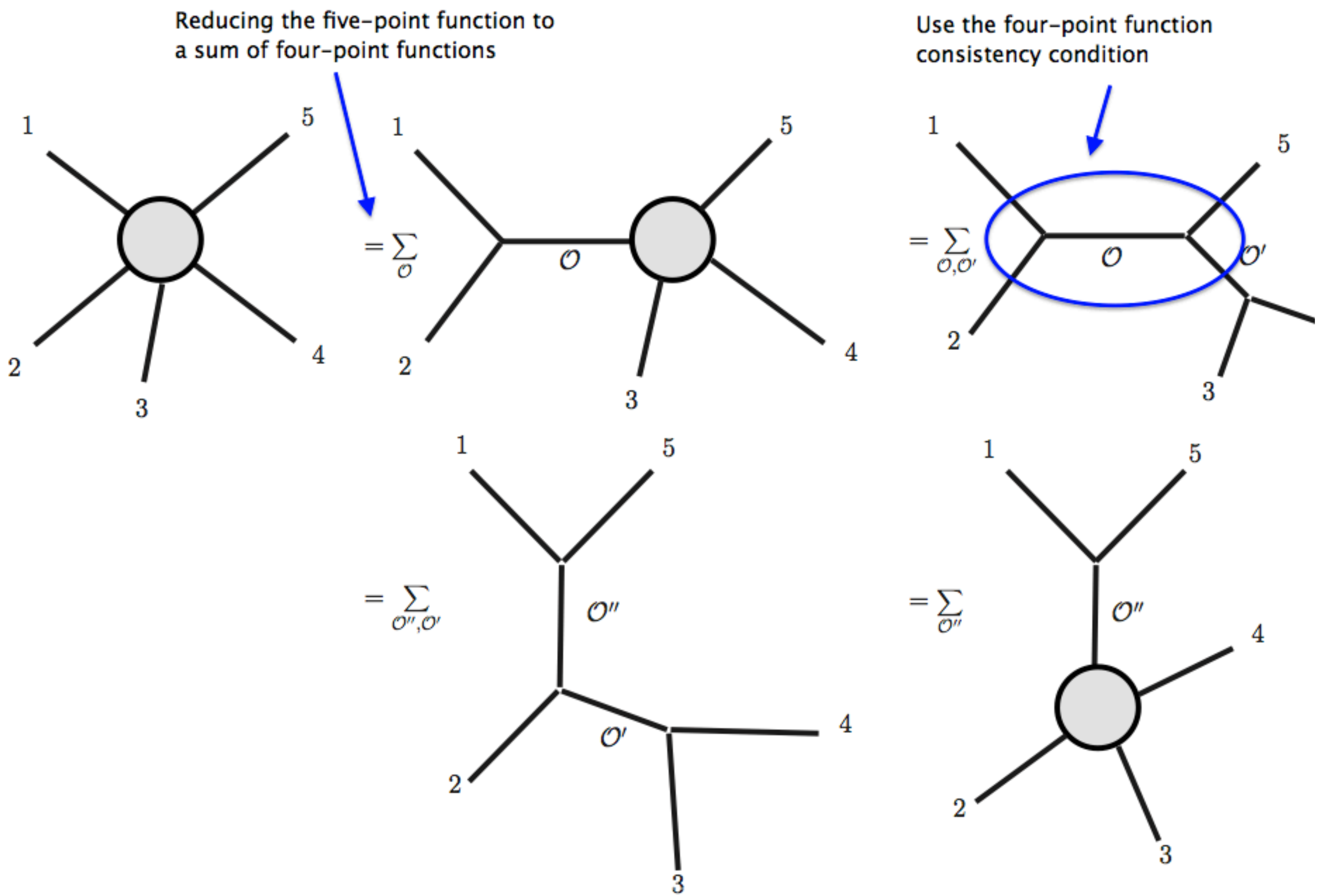
- (Almost) definition of what CFT is

- Should have isolated solutions of finite-dim. families corresponding to actual CFTs

- Many applications in D=2 [Belavin-Polyakov-Zamolodchikov'83]...

- Until recently thought useless/intractable in D>=3

5- and higher point functions don't give new constraints



D=2 success story

- In D=2 $(P_\mu, K_\mu, M_{\mu\nu}, D) \rightarrow$ Virasoro algebra

$\Rightarrow$ New lowering operators L_{-n}, $n=2,3,\ldots$

Virasoro multiplet = $\bigoplus_{n=1}^{\infty}$ (Conformal multiplets)

- Central charge c<1 + unitarity $\Rightarrow$

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, \ldots$$

[Friedan, Qiu, Shenker]

- Primary dimensions in these "minimal models" are also fixed:

$$\Delta_{r,s} = \frac{(r + m(r-s))^2 - 1}{2m(m+1)} \qquad 1 \le s \le r \le m-1$$

-Finally, knowing dimensions, OPE coefficients can be determined by **bootstrap**

[Belavin, Polyakov, Zamolodchikov], ...

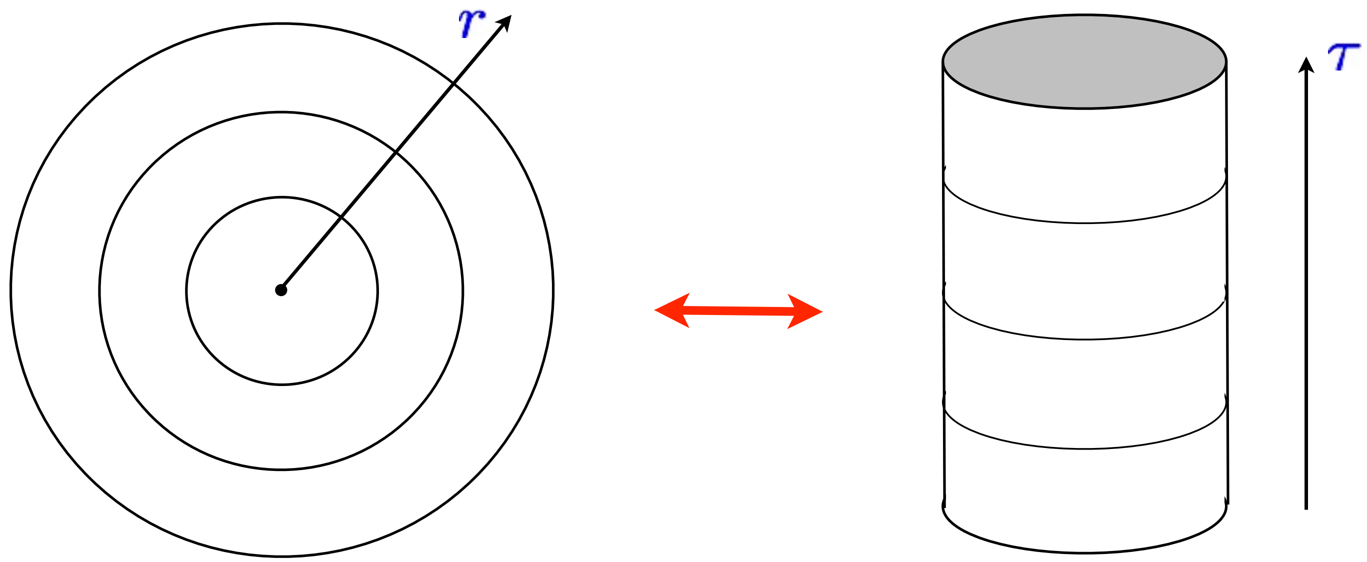
Difficulties in D>=3

$$\sum_i \lambda_{12i}\lambda_{34i}G(\Delta_i,\Delta_{ext}|u,v) = \sum_i \lambda_{14i}\lambda_{23i}G(\Delta_j,\Delta_{ext}|v,u)$$

- # of primaries is always infinite

- their dimensions are also unknowns to be computed

Asymptotics for the # of primaries (Cardy's formula in D dim's)



$S^{D-1} \times \mathrm{time}$

- Put the CFT to the sphere S^{D-1} of radius R x (time)
- via radial quantization, states on the sphere are in one-to-one correspondence with local operators in flat space $E_{sphere} = \dfrac{\Delta}{R}$

Now consider partition function: $Z = \sum \exp(-E/T)$

In the high T limit $T \gg R^{-1}$ expect:

$$Z \sim \exp(-\mathrm{vol}(S^{D-1})F(T)) \qquad F(T) = const.T^{D-1}$$

⇒ # of all states (primaries+descendants) should grow exponentially:

$$\#(\Delta < E) \sim \exp(Const.E^{1-1/D})$$

same is valid for the # of **primaries** (quasiprimaries if D=2)

Conf. block decomposition translated in radial quantization (useful for convergence)

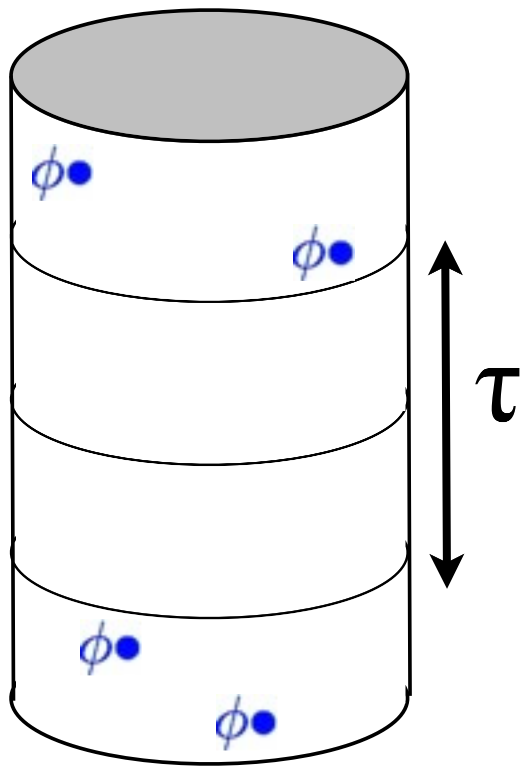
First translate OPE:

$$\phi(x)\phi(0) = \sum \lambda_{\mathcal{O}} C(x,\partial)\mathcal{O}(0)$$

$$\phi(0) \leftrightarrow |\Delta_\phi\rangle = \phi(0)|0\rangle$$

$$\phi(x)|\Delta_\phi\rangle = \sum \lambda_O \underbrace{(C_0(x)|\Delta_O\rangle + C_1(x)|\Delta_O + 1\rangle + \ldots)}_{\text{sum over descendants of O}}$$

$$\left\langle \begin{matrix} \phi(x_1)\ \phi(x_3) \\ \phi(x_2)\ \phi(x_4) \end{matrix} \right\rangle = \sum \lambda^2_{\phi\phi i} |x_{12}|^{\Delta_i - 2\Delta_\phi} |x_{34}|^{\Delta_i - 2\Delta_\phi} \langle \{O_i(x_2) + \cdots\}\{O_i(x_4) + \cdots\}\rangle$$



$$\langle \phi_3\phi_4\phi_1\phi_2\rangle \propto \sum \langle 0|\phi_3\phi_4|n\rangle e^{-E_n\tau}\langle n|\phi_1\phi_2|0\rangle$$

$$E_n = \Delta_{\mathcal{O}} + n, \ \ n = 0, 1, 2\ldots$$

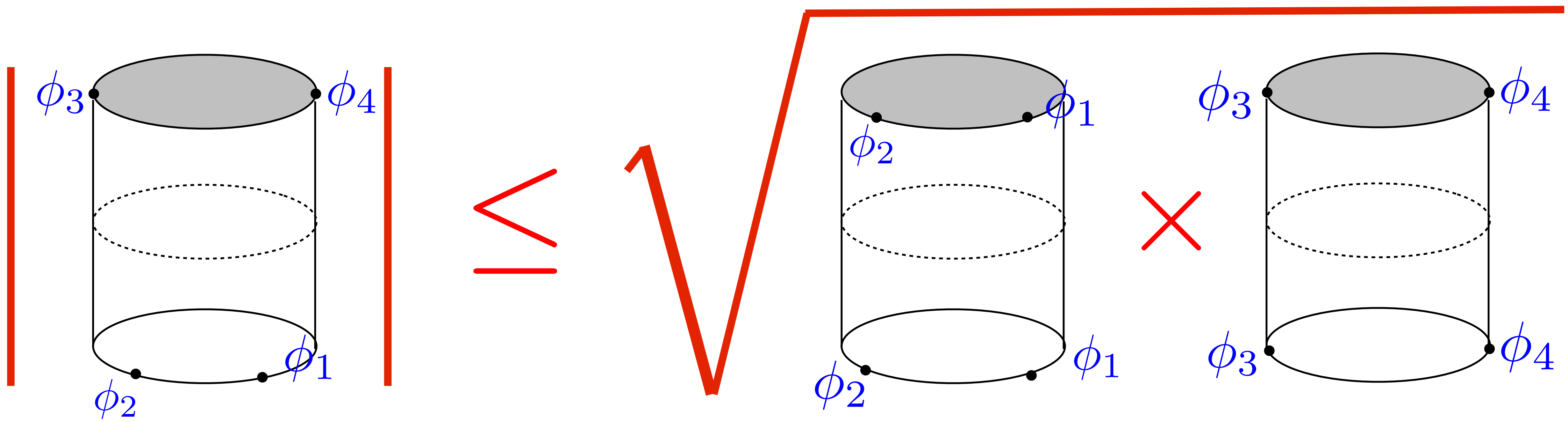
$$\phi \otimes \phi = \sum \mathcal{O}$$

Does this series converge?
It represents the 4-pt functions, so it better do...

But how fast?

Also, this series is not positive-definite; does it converge absolutely?

Cauchy inequality argument

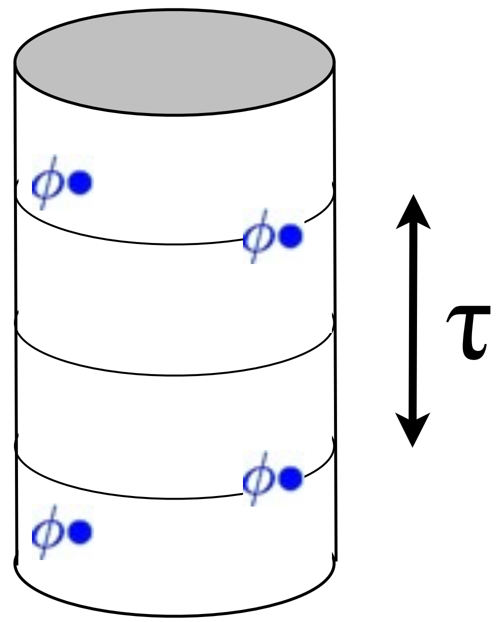


not reflection-positive

reflection positive
(norms of states)

Same inequality is valid for tails in OPE series

Convergence for reflection-positive 4-pt functions



$$\langle 0|\phi\phi\phi\phi|0\rangle = \sum_{E_n} \langle 0|\phi\phi|n\rangle e^{-E_n\tau}\langle n|\phi\phi|0\rangle$$

$$\langle 0|\phi\phi\phi\phi|0\rangle = \sum_{n} c_n^2 e^{-E_n\tau} \qquad c_n^2 \geq 0$$

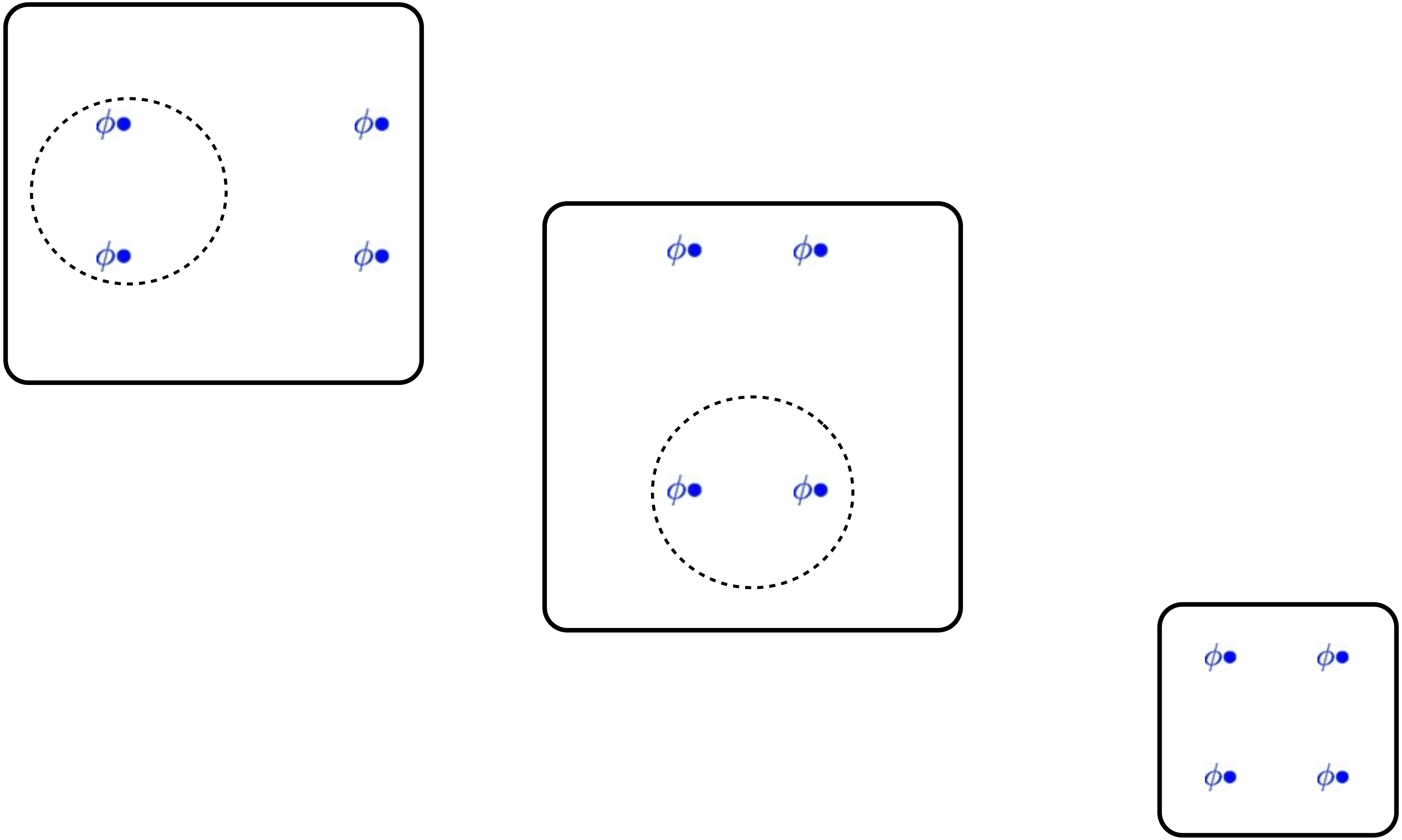
This series converges exponentially fast:

$$\sum_{E_n \geq E_*} c_n^2 e^{-E_n\tau} \lesssim e^{-E_*\tau}$$

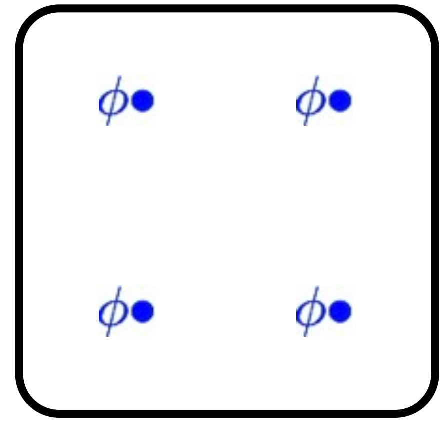
Rough proof: the only way this could not happen is if the coefficients c_n^2 grow exponentially with energy. But since in the small τ limit the correlator grows at most as a power of τ, c_n^2=O(E^some power)

In this last step it's important that $c_n^2 \geq 0$ and cancellations are impossible

Various geometries

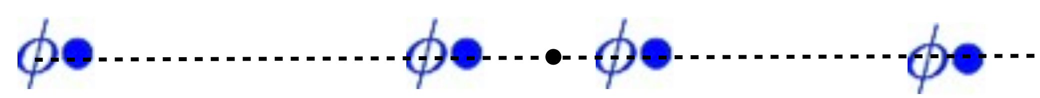


Democratic geometry

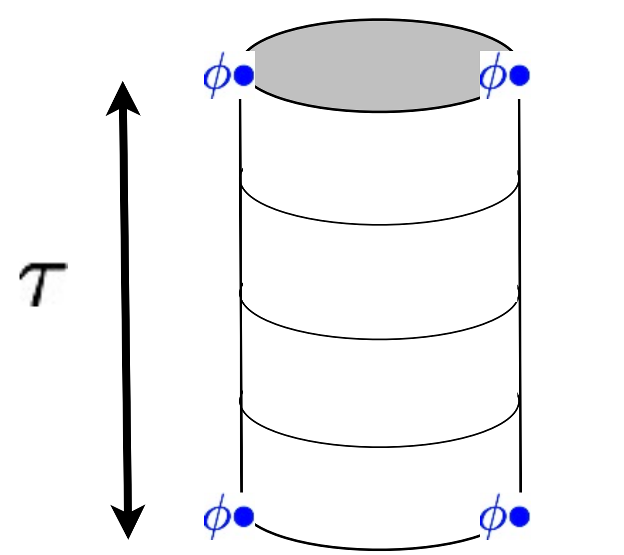


(near)-square configuration

conf.trans.



Weyl trans.



$S^{D-1} \times R$

$$e^{-\tau} = 3 - 2\sqrt{2} \approx 0.17$$

small parameter!

[Pappadopulo, S.R., Espin, Rattazzi‘2012]

Part 3
Conformal Bootstrap.
Concrete applications

Bootstrap applications for 3D Ising

Focus on 4-pt function of the spin field

$$\underbrace{\langle\sigma(x_1)\sigma(x_2)}\underbrace{\sigma(x_3)\sigma(x_4)}\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\sigma}|x_{34}|^{2\Delta_\sigma}}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

Applying OPE get:

$$g(u,v) = 1 + \sum f_i^2\, G_{\Delta_i,\ell_i}(u,v)$$

conformal blocks

$$\boxed{\begin{aligned}\sigma\times\sigma = 1 &+ (\epsilon + \epsilon' + \dots)\\ &+(T_{\mu\nu} + \dots)\\ &+(C_{\mu\nu\kappa\lambda} + \dots)\\ &+ \dots\end{aligned}}$$

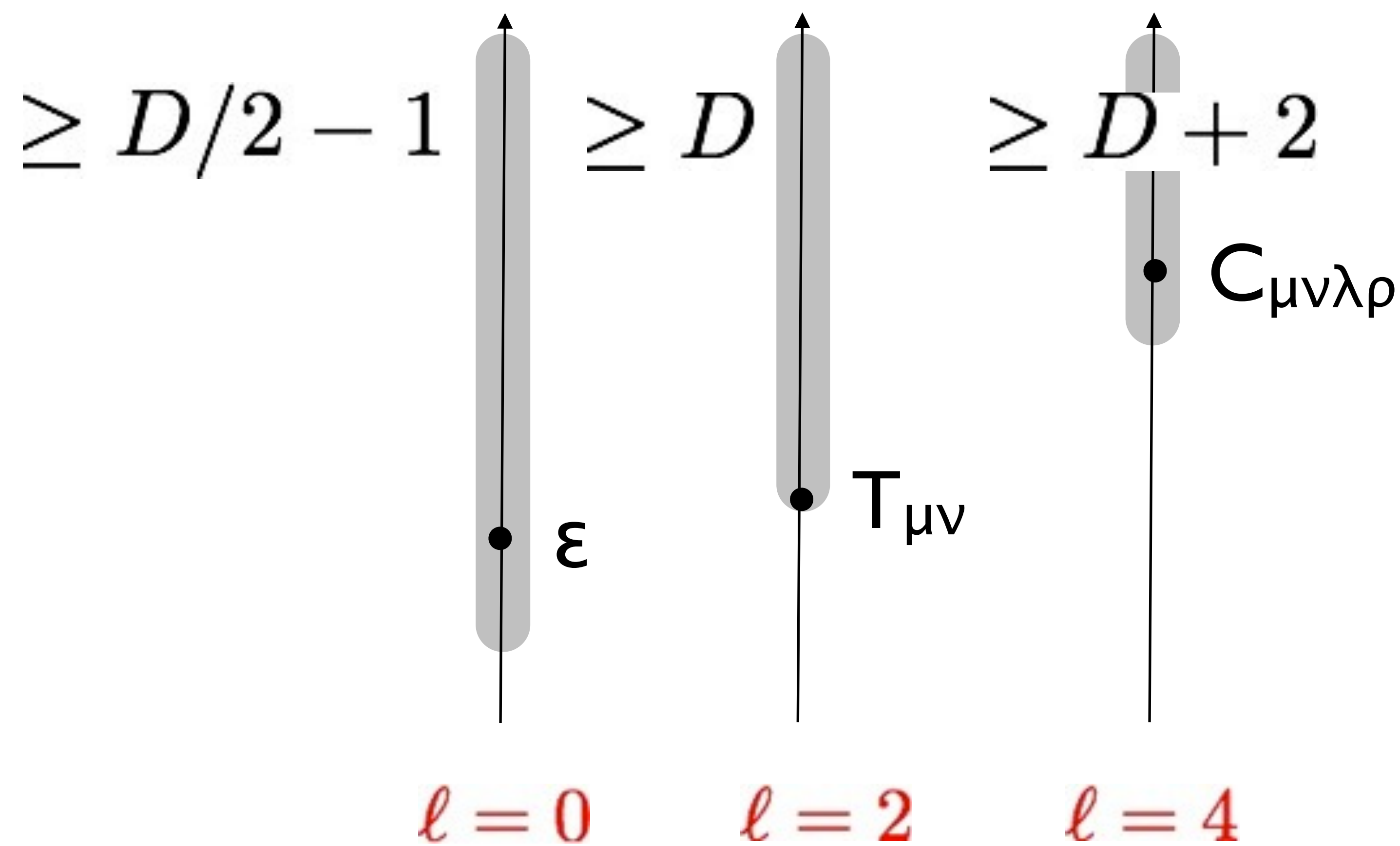
a priori, all fields above unitarity bounds are allowed

Crossing symmetry constraint:

$$\boxed{v^{\Delta_\sigma} g(u,v) = u^{\Delta_\sigma} g(v,u)}$$

Allowed vs realized spectrum in σ x σ OPE (D=3)

from Unitarity bounds:



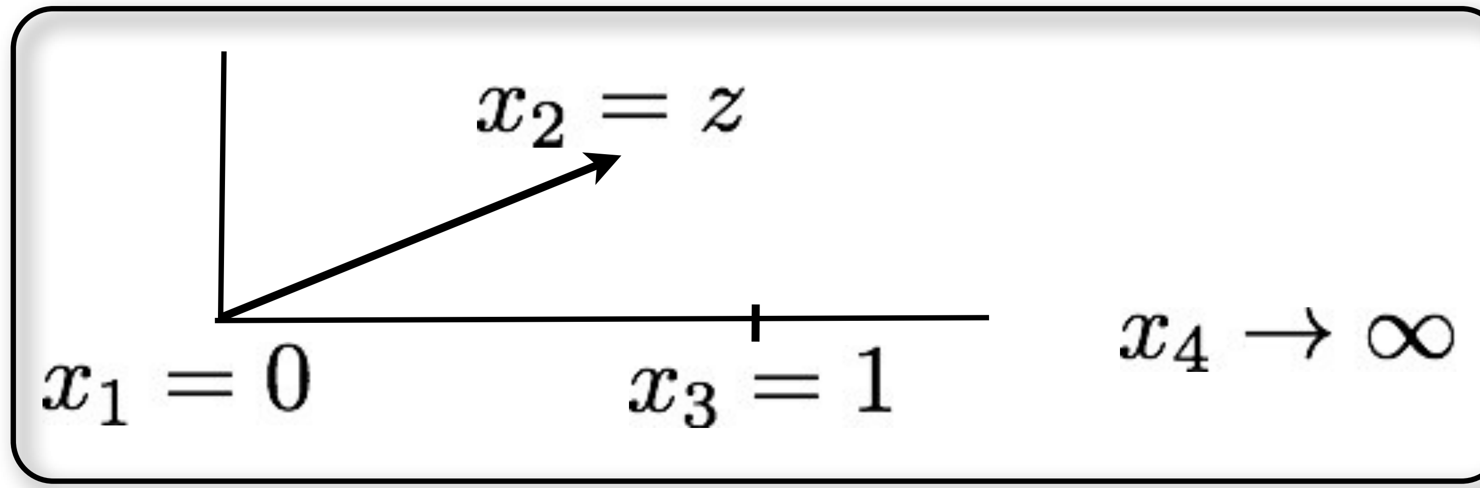
Theory of conformal blocks in D≥3

- **"simple" in even dims** [Dolan,Osborn,2001]

D=4: $$G_{\Delta,\ell}(u,v)=\frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z})-(z\leftrightarrow\bar{z})\right]$$

$$k_\beta(x)\equiv x^{\beta/2}{}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta;x\right)$$

$$u=z\bar{z},\qquad v=(1-z)(1-\bar{z})$$



- **Complicated in odd dims**

D=3: $$G_{\Delta,0}(u,v)=u^{\Delta/2}\sum_{m,n=0}^{\infty}\frac{[(\Delta/2)_m(\Delta/2)_{m+n}]^2}{m!\,n!(\Delta+1-\frac{D}{2})_m(\Delta)_{2m+n}}u^m(1-v)^n$$

+ recursions for higher spins

D=3 is as complicated as arbitrary D

Theory of conformal blocks in D≥3

- **Casimir differential equation** [Dolan,Osborn,2003]

$$\mathrm{Cas} \times G_{\Delta,l}(u,v) = [\Delta(\Delta - D) + l(l+D-2)]G_{\Delta,l}(u,v)$$

2nd order part.diff.op. coming from the quadratic Casimir

- **Approximate expressions** [Hogervorst,S.R.,to appear]

$$G_{\Delta,\ell}(u,v) \approx r^{\Delta} C_{\ell}^{(D/2-1)}(\cos\alpha) + \text{few } \%$$

$$re^{i\alpha} = \frac{z}{(1+\sqrt{1-z})^2}$$